Structure of $\mathscr{H}$-Classes in the Semigroup of Nonnegative Matrices

Yang Shangjun*
Department of Mathematics
University of Missouri — Kansas City
Kansas City, Missouri 64110

Submitted by George P. Barker

ABSTRACT

This paper investigates the structure of the $\mathscr{H}$-classes in the semigroup N_n of nonnegative matrices. We obtain two sets of equivalent conditions for any two matrices A, B to satisfy $A \mathscr{H} B$ in N_n. We establish a one-to-one and onto correspondence between the $\mathscr{H}$-class $\mathscr{H}_A$ and the group W_{A_0} of the greatest cone independent submatrix A_0 of A. We find W_{A_0} can be made up from the groups of the connective submatrices of A_0.

1. INTRODUCTION

The importance of Green's relations in the theory of nonnegative matrices is well known. Berman and Plemmons [1] and Tam [2] give a lot of important results on Green's relations, especially on the structure of $\mathscr{R}$-classes and $\mathscr{D}$-classes.

In this paper we shall concentrate on the structure of $\mathscr{H}$-classes. In Section 2 we investigate the cone linear independence properties of a nonnegative matrix and its submatrices in order to obtain some preparation for the group representation of $\mathscr{H}$-classes (Theorem 3.4). Theorems 3.1, 3.4, 3.9, 4.2, and 4.9 are the main results concerning the structure of $\mathscr{H}$-classes in N_n. The set $\{mA : m > 0\}$ consisting of all the positive multiples of the nonnegative matrix A is obviously an (in general, proper) subset of $\mathscr{H}_A$. Finding conditions under which the equality $\mathscr{H}_A = \{mA : m > 0\}$ holds is surely an interesting problem. Theorems 4.7, 4.13, and 4.14 answer this problem. The concept of connectivity introduced at the beginning of Section 4 plays an important role in that section.

*The permanent address of the author is Department of Mathematics, Anhui University, Hefei, Anhui, People's Republic of China.

52 Vanderbilt Ave., New York, NY 10017 0024-3795/84/$3.00

2. CONE LINEAR INDEPENDENCE

Let $A \in N_n$. We denote the polyhedral cone generated by all the columns [rows] of A by $G(A)$ $[\tilde{G}(A)]$, that is,

$$G(A) = AR_+^n \; \left[\tilde{G}(A) = R_+^n A\right];$$

here and in the following, we express the dual proposition in brackets.

A column [row] vector $x \in G(A)$ $[\tilde{G}(A)]$ is called extremal if for any $y \in G(A)[\tilde{G}(A)]$, $x - y \in G(A)$ $[\tilde{G}(A)]$ implies $y = cx$ for some $c \geqslant 0$. A column [row] of A is called extremal if it is an extremal vector of $G(A)$ $[\tilde{G}(A)]$.

According to the definition, a column [row] vector $x \in G(A)$ $[\tilde{G}(A)]$ is not extremal if and only if x can be expressed as

$$x = c_1 x_1 + c_2 x_2,$$

where $c_1, c_2 > 0$, $x_1, x_2 \in G(A)$ $[\tilde{G}(A)]$, and $x_1 \neq cx_2$ for any constant $c \geqslant 0$.

If $A \in N_n$, let A' $[\tilde{A}']$ denote an $n \times r$ $[s \times n]$ submatrix of A whose columns [rows] represent all the extreme rays of $G(A)$ $[\tilde{G}(A)]$. Then we have

$$G(A) = G(A') \qquad \left[\tilde{G}(A) = \tilde{G}(\tilde{A}')\right]. \tag{2.1}$$

We denote the number of columns [rows] of A' $[\tilde{A}']$ by $d(A)$ $[\tilde{d}(A)]$. It is clear that $A = 0$ if and only if $d(A) = 0$ $[\tilde{d}(A) = 0]$. Since we do not want the trivial case $A = 0$ to bother us, we shall assume in the following that $A \neq 0$ and thus that $d(A)$ and $\tilde{d}(A) = d(A^t)$ are positive.

PROPOSITION 2.1. *If $G(A) = G(B)$ $[\tilde{G}(A) = \tilde{G}(B)]$, then we have*

$$d(A) = d(B) \qquad \left[\tilde{d}(A) = \tilde{d}(B)\right]$$

and

$$A' = B'M \qquad \left[\tilde{A}' = M\tilde{B}'\right],$$

where $m \in N_n$ is $d(A)$-monomial $[\tilde{d}(A)$-monomial$]$.

The following theorem is a known result: (i) $\Leftrightarrow$ (ii) is proved in Tam [2, Theorem 6.1], and (i) $\Leftrightarrow$ (iii) is proved in Berman and Plemmons [1, (3.4.15) Theorem]. Here we put them together.

THEOREM 2.2. *The following are equivalent*:

(i) $A \mathscr{R} B$ $[A \mathscr{L} B]$;

(ii) $G(A)=G(B)$ $[\tilde{G}(A)=\tilde{G}(B)]$;

(iii) *there exists a* $d(A)$*-monomial* $[\tilde{d}(A)$*-monomial*$]$ *nonnegative matrix* M *such that*

$$A'=B'M \qquad [\tilde{A}'=M\tilde{B}'].$$

COROLLARY 2.3. *If* $A \mathscr{R} B$ $[A \mathscr{L} B]$ *in* N_n, *then*

$$d(A)=d(B) \qquad [\tilde{d}(A)=\tilde{d}(B)].$$

There is a misprint in [1, p. 71] concerning $d(XB)$; we state the correct relation in the next lemma.

LEMMA 2.4. *For any* $X, B \in N_n$, *we have*

$$d(XB) \leqslant d(B) \qquad [\tilde{d}(BX) \leqslant \tilde{d}(B)].$$

NOTE. It is not true, in general, that $d(BX) \leqslant d(B)$ $[\tilde{d}(XB) \leqslant \tilde{d}(B)]$.

COROLLARY 2.5. *If* $A \mathscr{L} B$ $[A \mathscr{R} B]$ *in* N_n, *then*

$$d(A)=d(B) \qquad [\tilde{d}(A)=\tilde{d}(B)].$$

PROPOSITION 2.6. *If* $A \mathscr{L} B$ $[A \mathscr{R} B]$, *then*

$$d(A)=d(B) \quad \textit{and} \quad d(A^t)=d(B^t)$$
$$[\tilde{d}(A)=\tilde{d}(B) \quad \textit{and} \quad \tilde{d}(A^t)=\tilde{d}(B^t)].$$

Proof. If $A \mathscr{L} B$ $[A \mathscr{R} B]$, then $A^t \mathscr{R} B^t$ $[A^t \mathscr{L} B^t]$, and thus the conclusion of this proposition follows immediately from the Corollaries 2.3 and 2.5. ∎

Using Proposition 2.6, we give a very simple proof of the following well-known result (cf. Berman and Plemmons [1, (3.4.13)]).

PROPOSITION 2.7. *If* $A \mathscr{D} B$ *in* N_n, *then*

$$d(A)=d(B) \quad \textit{and} \quad d(A^t)=d(B^t).$$

Proof. If $A \mathscr{D} B$, then $A \mathscr{R} C$ and $C \mathscr{L} B$ for some $C \in N_n$. By Proposition 2.6, we have

$$d(A) = d(C), \quad d(A^t) = d(C^t), \quad d(C) = d(B), \quad d(C^t) = d(B^t).$$

Hence

$$d(A) = d(C) = d(B) \quad \text{and} \quad d(A^t) = d(C^t) = d(B^t). \qquad \blacksquare$$

The converses of Corollaries 2.3 and 2.5 and Proposition 2.7 are not true. For example, the matrices

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

satisfy

$$d(A) = d(B) = 3 = d(A^t) = d(B^t)$$

and $A \not\mathscr{D} B$.

PROPOSITION 2.8. *If $A \in N_n$ is regular, then*

$$d(A) = d(A^t).$$

Proof. If $A = 0$, then $d(A) = d(A^t) = 0$. If rank $A = r > 0$, then by Theorem (4.9) of Berman and Plemmons [1], we have

$$A \mathscr{D} B, \qquad \text{where} \quad B = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $d(B) = d(B^t) = r$, we have, by Proposition 2.7, that

$$d(A) = r = d(A^t). \qquad \blacksquare$$

B. S. Tam [2] has given an example showing that $A \mathscr{J} B$ in N_n does not imply $d(A) = d(B)$. But in the case that A or B is regular, by using Corollary (6.4) of Tam [2], we can easily prove the following conclusion:

PROPOSITION 2.9. *If $A \mathscr{J} B$ in N_n and A or B is regular, then*

$$d(A) = d(B) = d(A^t) = d(B^t).$$

PROPOSITION 2.10. *For any permutation matrices P and Q, we have*

(i) $PAQ\mathscr{R}PBQ$ *iff* $A\mathscr{R}B$,
(ii) $PAQ\mathscr{L}PBQ$ *iff* $A\mathscr{L}B$,
(iii) $PAQ\mathscr{D}PBQ$ *iff* $A\mathscr{D}B$,
(iv) $PAQ\mathscr{H}PBQ$ *iff* $A\mathscr{H}B$.

Proof. (i) is true because the following statements are equivalent:

(a) $PAQ\mathscr{R}PBQ$,
(b) $(PAQ)X = PBQ$ and $(PBQ)Y = PAQ$ for $X, Y \in N_n$,
(c) $A(QXQ^t) = B$ and $B(QYQ^t) = A$,
(d) $A\mathscr{R}B$.

(ii) can be proved similarly.
(iii) and (iv) are direct corollaries of (i) and (ii). ∎

In the remainder of this paper, for simplicity, we use the following notation concerning a matrix A in N_n without further comment:

$$r = d(A), \qquad s = d(A^t),$$

A' = a submatrix consisting of r cone independent columns of A,
$\tilde{A}'$ = a submatrix consisting of s cone independent rows of A, and
A_0 = an $s \times r$ submatrix of A lying in A' and $\tilde{A}'$.

In addition, we call a matrix column [row] cone independent if all its columns [rows] are cone independent. We call a matrix cone independent if it is both column and row cone independent.

PROPOSITION 2.11. *If A_0 is an $s \times r$ submatrix of $A \in N_n$ lying in A' and $\tilde{A}'$, then A_0 is cone independent, or*

$$d(A_0) = d(A) = r \quad \text{and} \quad d(A_0^t) = d(A^t) = s.$$

Proof. Let

$$B = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}$$

be an $n \times n$ matrix; then

$$d(A_0) = d(B), \qquad d(A_0^t) = d(B^t).$$

Using Propositions 2.10 and 2.2, we have $A \mathcal{D} B$, from which the desired result follows. ■

The converse of Proposition 2.11 is also true, that is,

PROPOSITION 2.12. *If an $s \times r$ submatrix C of A is cone independent, where $r = d(A)$ and $s = d(A^t) = \tilde{d}(A)$, then the r columns of A belonging to C are cone independent, and the s rows of A belonging to C are cone independent.*

Proof. If these r columns [s rows] of A belonging to C are not cone independent, then all the columns [rows] of C are not cone independent either. ■

PROPOSITION 2.13. *Suppose $A \in N_n$ is partitioned as*

$$A = \begin{bmatrix} A_0 & A_1 \\ A_2 & A_3 \end{bmatrix}, \tag{2.2}$$

where A_0 is an $s \times r$ submatrix of A lying in A' and $\tilde{A}'$, then we have

$$A_1 = A_0 X, \tag{2.3}$$

$$A_2 = Y A_0, \tag{2.4}$$

$$A_3 = Y A_1 = A_2 X \tag{2.5}$$

for some $r \times (n-r)$ nonnegative matrix X and some $(n-s) \times s$ nonnegative matrix Y. Further, the matrix A_3 is independent of the matrices X and Y.

Proof. Since

$$A' = \begin{bmatrix} A_0 \\ A_2 \end{bmatrix}$$

contains $r = d(A)$ cone independent columns of A, there must be a $r \times (n-r)$

nonnegative matrix X such that

$$\begin{bmatrix} A_1 \\ A_3 \end{bmatrix} = A'X = \begin{bmatrix} A_0 X \\ A_2 X \end{bmatrix}.$$

Similarly, we have

$$[A_2, A_3] = Y[A_0, A_1] = [YA_0, YA_1]$$

for some $(n-s)\times s$ nonnegative matrix Y, and thus we have (2.3), (2.4), and (2.5).

If there is another $r\times(n-r)$ nonnegative matrix X_1 satisfying $A_0X_1 = A_1 = A_0X$, then we have

$$A_2X_1 = YA_0X_1 = YA_0X = A_2X.$$

Similarly, for any $(n-s)\times s$ nonnegative matrix Y_1, $Y_1A' = A_2 = YA'$ implies $Y_1A_1 = YA_1$. Therefore, if $A \in N_n$ is partitioned as in (2.2) with

$$A' = \begin{bmatrix} A_0 \\ A_2 \end{bmatrix}$$

and $\tilde{A}' = [A_0, A_1]$, then the matrix A_3 is uniquely determined by the matrices A_0, A_1, and A_2. ∎

In general, a matrix $A \in N_n$ cannot be partitioned as in (2.2), but we still have

THEOREM 2.14. *Any matrix A in N_n is uniquely determined by A' and $\tilde{A}'$. In other words, $A' = B'$ and $\tilde{A}' = \tilde{B}'$ imply $A = B$.*

Proof. First of all, we change the order of the columns and the rows of A to get a new matrix $\bar{A}$ in such a way that if $a_{i_1}, a_{i_2}, \ldots, a_{i_r}$ $(i_1 < i_2 < \cdots < i_r)$ belong to A', we let $\bar{a}_k = a_{i_k}$ for $k = 1,2,\ldots,r$, and keep the order of the other columns, that is, for $\bar{a}_s = a_{i_s}$ and $\bar{a}_t = a_{i_t}$, $s < t$ if and only if $i_s < i_t$; and the order of the rows of A is changed similarly. Thus we have

$$PAQ = \bar{A} = \begin{bmatrix} A_0 & \bar{A}_1 \\ \bar{A}_2 & A_3 \end{bmatrix}, \tag{2.6}$$

where P and Q are some permutation matrices and A_3 is the submatrix which we get by deleting A' and $\tilde{A}'$ from A. It is easy to see, from the construction of the matrix $\bar{A}$, that both $\bar{A}_1$ and $\bar{A}_2$ are uniquely determined by A' and $\tilde{A}'$.

By Equation (2.6) and Proposition 2.11, we have

$$d(\bar{A}) = d(A) = d(A_0),$$

$$d(\bar{A}^t) = d(A^t) = d(A_0^t).$$

and hence $\bar{A}$ is partitioned as in (2.2). Since $\bar{A}_1$ and $\bar{A}_2$ are uniquely determined by A' and $\tilde{A}'$, the matrix A_3 is uniquely determined by A' and $\tilde{A}'$, by Proposition 2.13. So is the matrix A. ∎

How can we determine A from A' and $\tilde{A}'$? First we find the P, Q, $\bar{A}_1$ and $\bar{A}_2$ of Equation (2.6). Then we find an $r \times (n-r)$ nonnegative matrix X such that $\bar{A}_1 = A_0 X$, and thus we find A_3 $(=\bar{A}_2 X)$ as well as $\bar{A}$. Finally, we get $A = P^t \bar{A} Q^t$.

3. THE STRUCTURE OF $\mathscr{H}$-CLASSES

We now investigate the structure of the $\mathscr{H}$-class in the semigroup N_n. First we establish the following important theorem, which is really a restatement of Theorem 2.2.

THEOREM 3.1. *The following statements are equivalent:*

(i) $A \mathscr{H} B$ *in* N_n,

(ii) $G(A) = G(B)$ *and* $\tilde{G}(A) = \tilde{G}(B)$ *(or* $G(A^t) = G(B^t)$*),*

(iii) *there exist monomial matrices* $M \in N_r$ *and* $N \in N_s$ *such that*

$$B' = A'M, \tag{3.1}$$

$$\tilde{B}' = N\tilde{A}', \tag{3.2}$$

$$B_0 = A_0 M = N A_0. \tag{3.3}$$

PROPOSITION 3.2. *Let* $A \mathscr{H} B$; *then the ith column [row] of* A *belongs to* A' $[\tilde{A}']$ *if and only if the ith column [row] of* B *belongs to* B' $[\tilde{B}']$.

Proof. It suffices only to prove the "only if" part. Suppose that the i_1th, i_2th,$\dots$,i_sth rows of A belong to $\tilde{A}'$ and the j_1th, j_2th,$\dots$,j_rth columns of A belong to A'. Then, by Theorem 3.1, the equations (3.1)–(3.3) hold for some monomial matrices M and N. By Equation (3.2), the submatrix consisting of the j_1th, j_2th,..., j_rth columns of $\tilde{B}'$ is NA_0; by Equation (3.1), the submatrix consisting of the i_1th, i_2th,..., i_sth rows of B' is A_0M; and by Equation (3.3), both these submatrices are equal to a cone independent $s \times r$ submatrix B_0 of B. Therefore the j_1th, j_2th,..., j_rth columns of B are cone independent and the i_1th, i_2th,..., i_sth rows of B are cone independent by Proposition 2.12. ■

If the matrix A is partitioned as in (2.2) (that is, its first r columns and first s rows are cone independent), then every matrix B in the $\mathscr{H}$-class $\mathscr{H}_A$ containing A must be partitioned as in (2.2) by Proposition 3.2. If A is not partitioned as in (2.2), we can find permutation matrices P and Q to partition the matrix $\bar{A} = PAQ$ as in (2.2). Then $\bar{B} \in \mathscr{H}_{\bar{A}}$ is also partitioned as in (2.2). By Propositions 2.10 and 3.2, $\bar{B} \in \mathscr{H}_{\bar{A}}$ if and only if $P^t\bar{B}Q^t \in \mathscr{H}_A$; and $B \in \mathscr{H}_A$ if and only if $PBQ \in \mathscr{H}_{\bar{A}}$. We may assume, without loss of generality, that A is partitioned as in (2.2), and hence every element in $\mathscr{H}_A$ is also partitioned as in (2.2).

Denote the set of all r-monomial matrices by

$$V_r = \{ M \in N_r : M \text{ is monomial} \}.$$

Then V_r is a group under matrix multiplication. Let G be the direct product of the groups V_r and V_s, that is,

$$G = V_r \times V_s = \{ (M, N) : M \in V_r, N \in V_s \},$$

with the multiplication as

$$(M_1, N_1) \cdot (M_2, N_2) = (M_1M_2, N_1N_2).$$

Then $(G, \cdot)$ is a group.

THEOREM 3.3. *For any nonzero $s \times r$ matrix, the set*

$$W_{A_0} = \{ (M, N) \in G : A_0M = NA_0 \}$$

is a subgroup of G. (We call it the group of A_0.)

Proof. It is clear that the identity (I_r, I_s) of the group G is in W_{A_0}. If $(M, N) \in W_{A_0}$, then $M^{-1} \in V_r$, $N^{-1} \in V_s$, and $A_0M = NA_0$ implies

$N^{-1}A_0 = A_0M^{-1}$, which means that $(M, N)^{-1} = (M^{-1}, N^{-1})$ is in W_{A_0}. Finally, if (M_1, N_1) and (M_2, N_2) are in W_{A_0}, then $M_1M_2 \in V_r$, $N_1N_2 \in V_s$, and $A_0M_1M_2 = N_1A_0M_2 = N_1N_2A_0$, which means that $(M_1, N_1)\cdot(M_2, N_2) = (M_1M_2, N_1N_2)$ is in W_{A_0}. Therefore W_{A_0} is a subgroup of G. ∎

The next theorem follows immediately from Theorems 2.14, 3.1, and 3.2.

THEOREM 3.4. *Let A', $\tilde{A}'$, A_0, and W_{A_0} be as before. Then*

$$\mathscr{H}_A = \left\{ B : B' = A'M,\ \tilde{B}' = N\tilde{A}',\ \text{and } (M, N) \in W_{A_0} \right\},$$

and the mapping $\phi : W_{A_0} \to \mathscr{H}_A$ with $\phi(M, N) = B$ is one-to-one and onto.

PROPOSITION 3.5. *Let*

$$C = \begin{bmatrix} A & AU \\ VA & VAU \end{bmatrix},$$

where $A \in N_n$, U is an $n \times k$ nonnegative matrix and V is an $l \times n$ nonnegative matrix. Then

$$C_0 = A_0 \tag{3.4}$$

and

$$\mathscr{H}_C = \left\{ \begin{bmatrix} B & BU \\ VB & VBU \end{bmatrix} : B \in \mathscr{H}_A \right\}. \tag{3.5}$$

Proof. Using the argument in the proof of Proposition 2.11, we have

$$d(C) = d(A) \quad \text{and} \quad d(C^t) = d(A^t),$$

from which (3.4) follows. (3.5) is easy to prove. ∎

If A has another $r \times s$ submatrix A_{0*} lying in cone independent columns and rows, then, since $A\mathscr{H}A$ with these two $r \times s$ submatrices A_0 and A_{0*}, we have, by Theorem 3.1, that $A_0 = A_{0*}K = LA_{0*}$ for some $(K, L) \in V_r \times V_s$. On the one hand, for any $(M, N) \in W_{A_0}$, we have

$$A_{0*}KM = A_0M = NA_0 = NLA_{0*},$$

which implies that $(KM, NL) \in W_{A_{0*}}$. On the other hand, for any $(M^*, N^*) \in W_{A_{0*}}$,

$$A_0 K^{-1} M^* = A_{0*} M^* = N^* A_{0*} = N^* L^{-1} A_0$$

implies that $(K^{-1}M^*, N^*L^{-1}) \in W_{A_0}$ or that $(M^*, N^*) = (KM, NL)$ for some $(M, N) \in W_{A_0}$. So we have proved the following result.

PROPOSITION 3.6. *If A_{0*} is another $s \times r$ cone independent submatrix of the matrix A such that $A_0 = A_{0*}K = LA_{0*}$, then*

$$W_{A_{0*}} = \{(KM, NL) : (M, N) \in W_{A_0}\}.$$

Further, the mapping $\Psi : W_{A_0} \to W_{A_{0*}}$ with $\Psi(M, N) = (KM, NL)$ is one-to-one and onto.

NOTE: Ψ is not a group isomorphism unless K and L are identity matrices.

It is easy to see that the group of I_r is $W_{I_r} = \{(M, M) : M \in V_r\}$. Thus for the matrix

$$\bar{A} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

we have, by Theorem 3.4, that

$$\mathscr{H}_{\bar{A}} = \left\{ \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} : M \in V_r \right\}.$$

Using this fact together with Proposition 2.10, we immediately get the following known result (see Robinson [3, Theorem 5]).

PROPOSITION 3.7. *Let E_r be an idempotent in N_n of rank r such that there exist permutation matrices P and Q with the property that*

$$A = PE_rQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $B \mathscr{H} E_r$ if and only if

$$PBQ = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix},$$

where $M \in V_r$.

If E is a general idempotent matrix in N_n of rank r, then, by a well-known theorem (Berman and Plemmons [1, (3.3.1) Theorem]), we have

$$E = PEP^t = \begin{bmatrix} J & JU & 0 & 0 \\ 0 & 0 & 0 & 0 \\ VJ & VJU & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.6}$$

$$J = \operatorname{diag}(T_1, T_2, \dots, T_r), \tag{3.7}$$

where P is a permutation matrix, T_i are positive idempotent matrices of rank 1, and U, V are arbitrary nonnegative matrices of appropriate sizes.

Equations (3.6) and (3.7) imply that $d(E) = d(E^t) = r$ and that $E_0 = \operatorname{diag}(t_1, t_2, \dots, t_r)$, where t_i are any elements of T_i, $i = 1, 2, \dots, r$. In this case the group of E_0 is

$$W_{E_0} = \{(M, N) : M, N \in V_r \text{ such that } E_0 M = N E_0\}.$$

Let $B_i \in \mathscr{H}_E$ correspond to $(M_i, N_i) \in W_{E_0}$, $i = 1, 2$. Then by Proposition 3.5 we have

$$B_i = \begin{bmatrix} J_i & J_i U & 0 & 0 \\ 0 & 0 & 0 & 0 \\ VJ_i & VJ_i U & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where J_i is an element of $\mathscr{H}_J$ corresponding to $(M_i, N_i) \in W_{E_0}$. Since every monomial matrix can be expressed in two ways as a product of a diagonal matrix and a permutation matrix, we can assume that

$$M_i = \operatorname{diag}(m_{i1}, m_{i2}, \dots, m_{ir}) P_i = P_i^* \operatorname{diag}(m_{i1}^*, m_{i2}^*, \dots, m_{ir}^*),$$

where m_{ij}, m_{ij}^* are positive numbers, and P_i, P_i^* are permutation matrices $i = 1, 2$. [We can easily prove that $P_i^* = P_i$, $\operatorname{diag}(m_{i1}^*, \dots, m_{ir}^*) = P_i^t \operatorname{diag}(m_{i1}, \dots, m_{ir}) P_i$.] The matrices J_i then can be respectively expressed as

$$\begin{aligned} J_i &= \operatorname{diag}(m_{i1} T_1, \dots, m_{ir} T_r) P_i(I_1, \dots, I_r) \\ &= P_i^*(I_1, \dots, I_r) \operatorname{diag}(m_{i1}^* T_1, \dots, m_{ir}^* T_r), \end{aligned}$$

where $P_i(I_1, \dots, I_r)$ is the same permutation matrix as P_i, but its elements are the identities $I_1,\dots,I_r$ satisfying $\operatorname{order}(I_1)=\operatorname{order}(T_1),\dots,$ $\operatorname{order}(I_r)=\operatorname{order}(T_r)$.

Let us now observe the element $J_{12}\in\mathscr{H}_J$ corresponding to $(M_1,N_1)(M_2,N_2)=(M_1M_2,N_1N_2)\in W_{E_0}$. Since

$$\begin{aligned} M_1M_2 &= P_1^*\operatorname{diag}(m_{11}^*,\dots,m_{1r}^*)\operatorname{diag}(m_{21},\dots,m_{2r})P_2 \\ &= P_1^*\operatorname{diag}(m_{11}^*m_{21},\dots,m_{1r}^*m_{2r})P_2 \end{aligned}$$

and $T_i^2=T_i$, a direct computation shows that $J_{12}=J_1J_2$ and further that $B_1B_2=B_{12}$, where $B_{12}\in\mathscr{H}_{\bar{E}}$ corresponds (M_1M_2,N_1N_2). So the bijective $\phi:W_{E_0}\to\mathscr{H}_{\bar{E}}$ with $\phi(M,N)=B$ is an isomorphism. Since W_{E_0} is a group, $\mathscr{H}_{\bar{E}}$ is also a group (see, for example, Fraleigh [5, Theorem 7.1]). Because $B\in\mathscr{H}_{\bar{E}}$ iff $P^tBP\in\mathscr{H}_E$ and $P^tB_1B_2P=(P^tB_1P)(P^tB_2P)$, the bijective $\Psi:\mathscr{H}_{\bar{E}}\to\mathscr{H}_E$ with $\Psi(B)=P_tBP$ is an isomorphism. So we have proved the following result:

THEOREM 3.8. *If E is an idempotent matrix of rank r in N_n, then $\mathscr{H}_E$ is a group under matrix multiplication. Further, the map $f:W_{E_0}\to\mathscr{H}_E$ with $f(M,N)=\Psi(\phi(M,N))$ is a group isomorphism.*

Conversely, for any $A\in N_n$, W_{A_0} is a group. Therefore, the map $\phi:W_{A_0}\to\mathscr{H}_A$ is a group isomorphism, which implies that $\mathscr{H}_A$ is a group and thus contains an idempotent matrix, the identity of $\mathscr{H}_A$. At last we have

THEOREM 3.9. *If $A\in N_n$ is of rank r, then the following statements are equivalent:*

(1) *$\mathscr{H}_A$ is a group under the matrix multiplication,*
(2) *the map $\phi:W_{A_0}\to\mathscr{H}_A$ with $\phi(M,N)=B$ is an isomorphism,*
(3) *$\mathscr{H}_A$ contains an idempotent matrix of rank r,*
(4) *there exist positive numbers $m_1,m_2,\dots,m_r$ and permutation matrices P_1, Q_1, and P such that*

$$P_1AQ_1=\begin{bmatrix} J & JU & 0 & 0 \\ 0 & 0 & 0 & 0 \\ VJ & VJU & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$J=\operatorname{diag}(m_1T_1,m_2T_2,\dots,m_rT_r)P(I_1,I_2,\dots,I_r),$$

where U,V,T_i are the same as in Equations (3.6) *and* (3.7).

NOTE. (3) $\Rightarrow$ (1) is a well-known result in the theory of semigroups (see Clifford and Preston [4, Theorem 2.16]). Here we have got it and other results by using a simple elementary method.

4. CONNECTIVE MATRICES

Let $A \in R^{m\times n}$. Connect any two nonzero entries of A with a horizontal line if they are in the same row or with a vertical line if in the same column. Thus two nonzero entries a_{ij} and a_{pq} are connected in A if there is a sequence of pairs of indices

$$(i,j)=(i_1,j_1),(i_2,j_2),\dots,(i_{s-1},j_{s-1}),(i_s,j_s)=(p,q)$$

such that $a_{i_rj_r} \neq 0$ for $r = 1,2,\dots,s$ and for any successive pair $(i_r,j_r),(i_{r+1},j_{r+1})$ either $i_r = i_{r+1}$ or $j_r = j_{r+1}$. The matrix A is called connective if any two nonzero entries of A are connected in A.

For any two nonzero entries a_{ij} and a_{pq} of A, we define a relation $\sim$ such that

$$a_{ij} \sim a_{pq} \quad \text{iff} \quad a_{ij} \text{ and } a_{pq} \text{ are connected in } A.$$

Now the following lemma is obvious.

LEMMA 4.1. *The relation $\sim$ is an equivalence relation on the set S_A of all nonzero entries of A.*

THEOREM 4.2. *For any matrix $A \in R^{m\times n}$, there exist two permutation matrices P and Q such that*

$$PAQ = \operatorname{diag}(A_1, A_2,\dots,A_k;0), \tag{4.1}$$

where $A_1, A_2,\dots,A_k$ are connective submatrices of A and the zero matrix 0 may not appear in some cases.

Proof. Let $S_A/\sim\ = \{G_1,G_2,\dots,G_k\}$ be the quotient set of S_A relative to the equivalence relation $\sim$. For each G_t $(t=1,2,\dots,k)$ and the two sets of indices

$$\{i_1,i_2,\dots,i_u\} = \{i: a_{ij} \in G_t\}$$

and

$$\{j_1, j_2, \ldots, j_v\} = \{j : a_{ij} \in G_t\},$$

let A_t be the $u \times v$ submatrix of A with respect to these u rows and these v columns. Then (4.1) follows directly from the fact that $S_A = G_1 \cup G_2 \cup \cdots \cup G_k$ and $G_i \cap G_j = \varnothing$ $(i \neq j)$. ∎

PROPOSITION 4.3. *For any nonnegative matrix A we have:*

(i) $d(A) = 1$ *iff* $d(A^t) = 1$. *In this case A is connective.*

(ii) $d(A) = 2$ *iff* $d(A^t) = 2$. *In this case* A_0 *is connective iff* A_0 *has exactly one or no zero entry.*

Proof. (i) is immediate.

(ii): If $d(A) = 2$, then $d(A^t) \geqslant 2$ by (i). Suppose that $d(A^t) > 2$ and that the first three rows of A_0 are (a_{11}, a_{12}), (a_{21}, a_{22}), and (a_{31}, a_{32}), then the linear system

$$a_{11}x_1 + a_{21}x_2 + a_{31}x_3 = 0,$$
$$a_{12}x_1 + a_{22}x_2 + a_{32}x_3 = 0$$

has a solution as

$$x_1 : x_2 : x_3 = \begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix} : \begin{vmatrix} a_{31} & a_{11} \\ a_{32} & a_{12} \end{vmatrix} : \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}.$$

These three x's should be all nonzero, and one of them, say x_3, should have its sign different from the others'. Further we have

$$(a_{31}, a_{32}) = c_1(a_{11}, a_{12}) + c_2(a_{21}, a_{22}),$$

which is contrary to Theorem 2.11. Therefore $d(A^t) = 2$. The rest of (ii) is clear. ∎

THEOREM 4.4. *If* $3 = d(A^t) < d(A)$ *or* $3 = d(A) < d(A^t)$, *then* A_0 *is connective.*

Proof. If A_0 were not connective, then A_0 would have, by Theorem 4.2, a connective submatrix having two rows (columns) and more than two columns (rows), which is contrary to Proposition 4.3(ii). ∎

COROLLARY 4.5. *If* $4 = d(A^t) < d(A)$ *or* $4 = d(A) < d(A^t)$, *then either* A_0 *is connective or* A_0 *is a direct sum of a* 1×1 *submatrix and a connective submatrix.*

It is clear that multiplying a matrix by a monomial matrix does not change its connectivity. So the next proposition follows from Theorem 3.1.

PROPOSITION 4.6. *Let* A_0 *and* A_{0*} *be two* $s \times r$ *submatrices of* $A \in N_n$ *lying in cone independent columns and rows. Then* A_0 *is connective if and only if* A_{0*} *is connective.*

Now let $A = [a_{ij}] \in N_n$. The set $\{mA : m > 0\}$ is obviously a subset of $\mathscr{H}_A$, or equivalently, the set

$$I_d = \{(mI_r, mI_s) : m > 0\}$$

is a subset of W_{A_0}. In addition, since for any positive numbers m and l, $(mI_r, mI_s)^{-1} = (m^{-1}I_r, m^{-1}I_s) \in I_d$ and $(mI_r, mI_s)\cdot(lI_r, lI_s) = (mlI_r, mlI_s) \in I_d$, the set I_d is also a subgroup of the group W_{A_0}. So it may happen that $W_{A_0} = I_d$. And this is just the necessary and sufficient condition for $\mathscr{H}_A$ to be equal to the set $\{mA : m > 0\}$, by the following proposition.

PROPOSITION 4.7. $\mathscr{H}_A = \{mA : m > 0\}$ *if and only if* $W_{A_0} = \{(mI_r, mI_s) : m > 0\} = I_d$ *for some cone independent* $s \times r$ *submatrix* A_0 *of* A.

Proof. We know, by Theorem 3.4, that

$$\mathscr{H}_A = \left\{ B : B' = A'M, \tilde{B}' = N\tilde{A}', (M, N) \in W_{A_0} \right\}$$

and the map $\phi : W_{A_0} \to \mathscr{H}_A$ with $\phi(M, N) = B$ is one-to-one and onto. So it follows immediately that $\mathscr{H}_A = \{mA : m > 0\}$ iff $W_{A_0} = I_d$. ∎

COROLLARY 4.8. *If any cone independent* $s \times r$ *submatrix* A_0 *of the nonnegative matrix* A *has its group* $W_{A_0} = I_d$, *then every cone independent* $s \times r$ *submatrix* A_{0*} *of* A *has its group* $W_{A_{0*}} = I_d$.

THEOREM 4.9. *Let* A_0 *be an* $s \times r$ *cone independent submatrix of* $A \in N_n$. *By Proposition* 2.10 *and Theorem* 4.2, *we may assume, without loss of* *generality, that*

$$A_0 = \operatorname{diag}(A^{(1)}, A^{(2)}, \dots, A^{(k)}), \tag{4.2}$$

where $A^{(i)}$ $(i = 1, 2, \dots, k)$ *are connective submatrices of* A_0. *Then*

$$W_{A_0} = G_1 = \{(\operatorname{diag}(c_1 M^{(1)}, \dots, c_k M^{(k)}), \operatorname{diag}(c_1 N^{(1)}, \dots, c_k N^{(k)})) :$$

$$(M^{(i)}, N^{(i)}) \in W_{A^{(i)}},\ c_i > 0\}, \tag{4.3}$$

if for any $i \neq j$ $(1 \leqslant i, j \leqslant k)$, *there are no monomial matrices* H_1, H_j *such that*

$$A^{(i)} H_i = H_j A^{(j)}. \tag{4.4}$$

Proof. For any $(M, N) \in W_{A_0}$, we have positive numbers $m_1, \dots, m_r$, $n_1, \dots, n_s$ such that

$$A_0 M = N A_0, \tag{4.5}$$

$$M = \operatorname{diag}(m_1, \dots, m_r) P = \operatorname{diag}(M_1, \dots, M_k) P, \tag{4.6}$$

$$N = Q \operatorname{diag}(n_1, \dots, n_s) = Q \operatorname{diag}(N_1, \dots, N_k), \tag{4.7}$$

where P, Q are permutation matrices, and M_i (N_i) are block diagonal matrices with the numbers of their columns (rows) equal to the numbers of the columns (rows) of $A^{(i)}$, $i = 1, \dots, k$. By (4.2), (4.5)–(4.7), we have

$$\operatorname{diag}(A^{(1)} M_1, \dots, A^{(k)} M_k) P = Q \operatorname{diag}(N_1 A^{(1)}, \dots, N_k A^{(k)}) \tag{4.8}$$

or

$$\operatorname{diag}(A^{(1)} M_1, \dots, A^{(k)} M_k) = Q \operatorname{diag}(N_1 A^{(1)}, \dots, N_k A^{(k)}) P^{-1}. \tag{4.9}$$

We assert that the permutation $Q(P^{-1})$ does not exchange any row (column) belonging to $N_i A^{(i)}$ with any row (column) belonging to $N_j A^{(j)}$ if $i \neq j$. Suppose Q exchanges a row r_1 belonging to $N_i A^{(i)}$ with a row belonging to $N_j A^{(j)}$ $(i \neq j)$. Since $N_i A^{(i)}$ is connective, r_1 should have a nonzero entry belonging to a column l_1 of $N_i A^{(i)}$ which has one nonzero entry not belonging to r_1. Owing to (4.9), the permutation P^{-1} should exchange l_1 with a column belonging to $N_j A^{(j)}$. And l_1 should have a nonzero entry belonging to a row $r_2 \neq r_1$, while r_2 has one nonzero entry $\notin l_1$. Owing to (4.9) again, Q should exchange r_2 with a row belonging to $N_j A^{(j)}$. Then P^{-1} should exchange l_2 with a column belonging to $N_j A^{(j)}$, and so on. Therefore, Q (P^{-1}) must exchange all the rows (columns) belonging to $N_i A^{(i)}$ with all the rows (columns) belonging to $N_j A^{(j)}$. Hence Equation (4.9) implies that

$$A^{(i)} M_i = Q_j N_j A^{(j)} P_i^{-1},$$

where P_i and Q_j are some permutations. Let $H_i = M_i P_i$, $H_j = Q_j N_j$; then H_i and H_j are monomial satisfying (4.4), which is impossible.

Now we can write (4.8) as

$$\operatorname{diag}\left(c_1 A^{(1)} M^{(1)}, \dots, c_k A^{(k)} M^{(k)}\right) = \operatorname{diag}\left(c_1 N^{(1)} A^{(1)}, \dots, c_k N^{(k)} A^{(k)}\right),$$

where c_i are arbitrary positive numbers and $M^{(i)}$, $N^{(i)}$ are monomial matrices satisfying

$$A^{(i)} M^{(i)} = N^{(i)} A^{(i)}, \qquad i = 1, \dots, k. \qquad \blacksquare$$

COROLLARY 4.10. *If the matrix A_0 satisfies the conditions of Theorem* 4.9, *but* (4.4) *holds for some $i \neq j$, then*

$$W_{A_0} = G_1 \cup G_2,$$

where G_1 is as in (4.2) *and G_2 contains all possible elements as*

$$\left(\begin{bmatrix} c_1 M^{(1)} & & & & & \\ & \ddots & & & & \\ & & 0 & cH_i & & \\ & & & \ddots & & \\ & & cH_i^{-1} & 0 & & \\ & & & & \ddots & \\ & & & & & c_k M^{(k)} \end{bmatrix},\right.$$

$$\left.\begin{bmatrix} c_1 N^{(1)} & & & & & \\ & \ddots & & & & \\ & & 0 & cH_j & & \\ & & & \ddots & & \\ & & cH_j^{-1} & 0 & & \\ & & & & \ddots & \\ & & & & & c_k N^{(k)} \end{bmatrix}\right).$$

COROLLARY 4.11. *If A_0 is not connective, then $W_{A_0} \neq I_d$.*

By Theorem 4.9 and Corollary 4.10, the group of A_0 can be made up from the groups of its connective submatrices. So we may confine ourselves to the groups of connective matrices.

THEOREM 4.12. *If A_0 is connective and all its nonzero entries are the same, then*

$$W_{A_0} = \{(M, N) = m(Q, P) : m > 0\},$$

where Q and P are permutation matrices satisfying

$$A_0 Q = P A_0. \tag{4.10}$$

Proof. Let $M = Q\operatorname{diag}(m_1, \dots, m_r)$, $N = \operatorname{diag}(n_1, \dots, n_s)P$, and $S = A_0 Q = [s_{ij}]$. Then $A_0 M = N A_0$ implies

$$S\operatorname{diag}(m_1, \dots, m_r) = \operatorname{diag}(n_1, \dots, n_s) P S Q^t. \tag{4.11}$$

Hence $PSQ^t = S$, and then

$$[m_j s_{ij}] = [n_i s_{ij}], \qquad 1 \leqslant i \leqslant s, \quad 1 \leqslant j \leqslant r,$$

or

$$[m_j e_{ij}] = [n_i e_{ij}], \qquad 1 \leqslant i \leqslant s, \quad 1 \leqslant j \leqslant r, \tag{4.12}$$

where

$$e_{ij} = \begin{cases} 1 & \text{if} \quad s_{ij} \neq 0, \\ 0 & \text{if} \quad s_{ij} = 0. \end{cases}$$

Since the matrix $E = [e_{ij}]$ is connective, (4.12) implies

$$m_1 = \cdots = m_r = n_1 = \cdots = n_s = m. \qquad \blacksquare$$

THEOREM 4.13. *If A_0 satisfies the conditions of Theorem 4.12, then $\mathscr{H}_A \neq \{mA : m > 0\}$ iff neither of the matrices P and Q satisfying (4.10) is the identity.*

Proof. If one of the matrices Q and P in (4.10) is not the identity, then the other is not either, because A_0 is row and column cone independent. In the case $W_{A_0} \neq I_d$, or, by Proposition 4.7, $\mathscr{H}_A \neq \{mA : m > 0\}$. ■

THEOREM 4.14. *Suppose $d(A) = 2$ and A_0 is connective. Then we have:*

(i) *Every element $(M, N) \in W_{A_0}$ satisfies*

$$M = mP, \qquad N = mQ,$$

where $m > 0$; Q and P are permutation matrices.

(ii) $W_{A_0} \neq I_d$ *iff*

$$A_0 = \begin{bmatrix} u & v \\ v & u \end{bmatrix} \gg 0, \qquad u \neq v.$$

Proof. If (M, N) is any element in W_{A_0}, then we have

$$M = Q \operatorname{diag}(m_1, m_2),$$

$$N = \operatorname{diag}(n_1, n_2) P,$$

and

$$S \operatorname{diag}(m_1, m_2) = \operatorname{diag}(N_1, N_2) PSQ^t, \tag{4.13}$$

where $m_i, n_i > 0$, P, Q are permutation matrices, and $S = A_0 Q$ is connective. We have three cases:

(a) $P = Q = I_2$. In this case (4.13) implies

$$m_1 = m_2 = n_1 = n_2 = m, \tag{4.14}$$

which means $M, N \in I_d$.

(b) P or Q is I_2, but the other is not. In this case (4.13) gives

$$s_{11} : s_{21} = s_{12} : s_{22},$$

which contradicts the fact that $d(S) = d(A_0) = 2$. So case (b) is impossible.

(c) $P = Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. If A_0 has only one zero entry, then (4.13) can not hold. If A_0 is positive, so is S. In this case (4.13) still implies (4.14), and thus

$$M = N = \begin{bmatrix} 0 & m \\ m & 0 \end{bmatrix} \notin I_d.$$

If we denote $A_0 = [a_{ij}]$, then $A_0 M = N A_0$ becomes

$$\begin{bmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix},$$

from which we get

$$a_{11} = a_{22} = u > 0 \quad \text{and} \quad a_{12} = a_{21} = v > 0. \qquad \blacksquare$$

The author is thankful to Professor George P. Barker for making many suggestions and comments during the whole work.

REFERENCES

1 A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic, 1979.
2 B. S. Tam, A geometric treatment of generalized inverses and semigroups of nonnegative matrices, *Linear Algebra Appl.* 41:225–272 (1981).
3 C. E. Robinson, Jr., Green's relations on the semigroup of nonnegative matrices, M.S. thesis, Auburn Univ., 1978.
4 A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Vol. 1, Math. Surveys No. 7, Amer. Math. Soc., Providence, 1961.
5 John B. Fraleigh, *A First Course in Abstract Algebra*, Addison-Wesley Series in Mathematics, 1976.

Received 24 January 1983; revised 15 April 1983